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One-dimensional Klein–Gordon equation with logarithmic nonlinearities

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Abstract

In this paper, we consider the relativistic version of the logarithmic Schrödinger equation, namely the logarithmic Klein–Gordon equation. We show the existence of classical solutions, and we investigate weak solutions. Finally, we examine the traveling waves of this model.

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1. Introduction

The logarithmic Schrödinger equation has been introduced by Białynicki-Birula and Mycielski (see [2, 3]) and has the form

$$\mathrm{i}\frac{\partial u(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta u(x,t) + V(x,t)u(x,t) - \varepsilon u(x,t)\log(\delta|u(x,t)|^2). \qquad (1)$$

The quantities ε and δ are the parameters, where ε measures the force of the nonlinear interaction (positive ε means the attraction). The parameter δ is needed in order to make the argument of the logarithm dimensionless.

This nonlinearity is selected by assuming the separability of noninteracting subsystems property (see [2, 3]). It means that a solution of the nonlinear equation for the whole system can be constructed, as in the linear theory, by taking the product of two arbitrary solutions of the nonlinear equations for the subsystems. Its most attractive features are existence of the lower energy bound and validity of Planck's formula $E = \hbar\omega$.

The Klein–Gordon equation with logarithmic potential has also been introduced in the quantum field theory by Rosen [23]. Such kinds of nonlinearity appear naturally in inflation cosmology and in supersymmetric field theories (see [1, 10, 19]).

The logarithmic quantum mechanics possess some special analytic solutions (see [4, 17, 21]). For example, this model has a large set of oscillating localized solutions. In the paper [4] the authors studied the so-called Gaussons. Gaussons represent solutions of the Gaussian shape. Moreover, the interaction of Gaussons was studied in [20]. Using the Bohr–Sommerfeld quantization of localized solutions the mass spectrum of the localized particle-like collective excitations has been found [5].

It has been shown experimentally (see [12, 24, 25]) that the nonlinear effects in the quantum mechanics are very small. The upper bound for the parameter ε has been estimated, namely $|\varepsilon| \leqslant 3.3 \times 10^{-15}$ eV. Still, this equation is applied in many branches of physics, e.g. nuclear physics, optics and geophysics (see, e.g., [6, 11, 16, 18]).

In the present paper we shall work with the relativistic version of equation (1), namely with the Klein–Gordon equation:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)u = \frac{m^2c^2}{\hbar^2}u + \varepsilon u \log|u|^2.$$

We put $c = 1, m = 1$ and $\hbar = 1$ for simplicity. We will be interested in the one-dimensional case. In this system of units our equation takes the form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)u = u + \varepsilon u \log|u|^2. \tag{2}$$

Our first goal is to prove the existence of weak solutions to the Cauchy problem

$$\begin{cases} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)u = u + \varepsilon u \log|u|^2, \\ u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{3}$$

Moreover, we prove the existence of classical solutions of (3). What is more, we also want to show the existence of traveling waves for (2).

Let us mention a few words about the methods applied in this paper. In order to establish the existence of weak solution to the problem (3), we construct an appropriate approximate sequence. Next, we show *a priori* estimates, and we pass to the limits. The existence and uniqueness of classical solutions will be shown by means of topological methods, namely, we introduce the appropriate function spaces and apply there the Banach fixed point theorem. In order to obtain the classical solution, we consider the initial data, which are distant enough from zero (in the meaning of the L^∞-norm)—we do so, because of the singularity of the function $x \log|x|$ at the point zero. Finally, we investigate the existence of traveling waves. We examine the phase portrait near the equilibrium points. Different methods are employed for different points—for the hyperbolic one we apply the Grobman–Hartman theorem, but in other cases we use the Morse lemma. Subsequently, using the fact that the vector field is the Hamiltonian one, we construct a curve, which connects the critical points.

Let us finally comment on the relevant mathematical literature. The evolution problem (2) in dimensions higher than one was treated mathematically by Cazenave and Haraux (see [7–9]).

In addition, the logarithmic Schrödinger equation was widely investigated in [7, 13, 14].

2. Weak solution

The process of multiplication of equation (3) by a test function ϕ and integration by parts leads us to

Definition 1. *We shall say that* $u \in L^\infty(0,T;H^1(\mathbb{R}))$ *is a weak solution to the problem (3) if and only if for any* $\phi \in C_0^\infty(\mathbb{R}\times(-\infty,T))$ *the following identity holds:*

$$\int_0^T\int_{\mathbb{R}} u(x,t)\phi_{tt}(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\phi_t(x,0)\,\mathrm{d}x - \int_{\mathbb{R}} u_1(x)\phi(x,0)\,\mathrm{d}x$$
$$+\int_0^T\int_{\mathbb{R}} \nabla u(x,t)\nabla\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_{\mathbb{R}} u(x,t)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\varepsilon\int_0^T\int_{\mathbb{R}} u(x,t)\log|u(x,t)|^2\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t, \qquad (4)$$

where $u_0, u_1 \in L^1_{loc}(\mathbb{R})$.

Then for these solutions, we can state

Theorem 1. *Suppose that* $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, $|u_0|^2\log|u_0| \in L^1(\mathbb{R})$ *and* $\varepsilon \in [-1,0]$. *Then there exists a weak solution u to the problem (3). Moreover,* $u_t \in L^\infty(0,T;L^2(\mathbb{R}))$.

Proof. First of all, let us fix the number $n \in \mathbb{N}$. Next, let us denote by $\mathcal{O}^n$ the following set

$$\mathcal{O}^n = [-n-|u_0(-n)|, n+|u_0(n)|].$$

Subsequently, we define u_0^n, u_1^n as follows:

$$u_0^n(x) = \begin{cases} \dfrac{u_0(-n)}{|u_0(-n)|}x + u_0(-n)\left(1+\dfrac{n}{|u_0(-n)|}\right) & \text{for}\quad x\in[-n-|u_0(-n)|,-n],\\ u_0(x) & \text{for}\quad x\in(-n,n),\\ -\dfrac{u_0(n)}{|u_0(n)|}x + u_0(n)\left(1+\dfrac{n}{|u_0(n)|}\right) & \text{for}\quad x\in[n,n+|u_0(n)|],\\ 0 & \text{for}\quad x\in\mathbb{R}\backslash\mathcal{O}^n. \end{cases}$$

$$u_1^n(x) = \begin{cases} u_1(x) & \text{for}\quad x\in\mathcal{O}^n,\\ 0 & \text{for}\quad x\in\mathbb{R}\backslash\mathcal{O}^n. \end{cases}$$

Let us note that $u_0^n \in H_0^1(\mathcal{O}^n)$. Next, by a straightforward calculation, we check

Lemma 1. *The following convergence holds:*

$$\left\|u_0^n - u_0\right\|_{H^1(\mathbb{R})} \longrightarrow 0, \qquad \left\|u_1^n - u_1\right\|_{L^2(\mathbb{R})} \longrightarrow 0,$$

as $n \longrightarrow \infty$.

Using the same methods as in [15], one can show the following result:

Proposition 1. *For each n, the following problem,*

$$\begin{array}{lll} f_{tt}^n - f_{xx}^n = -f^n - \varepsilon f^n\log|f^n|^2, & x\in\mathcal{O}^n, & t\in(0,T),\\ f^n(x,0) = u_0^n(x), \quad f_t^n(x,0) = u_1^n(x), & x\in\mathcal{O}^n, & \\ f^n(x,t) = 0, & x\in\partial\mathcal{O}^n, & t\in(0,T), \end{array} \qquad (5)$$

possesses a weak solution f^n *such that* $f^n \in L^\infty\left(0,T;H_0^1(\mathcal{O}^n)\right)$ *and* $f_t^n \in L^\infty(0,T;L^2(\mathcal{O}^n))$. *Moreover, the following estimate holds:*

$$\|f^n\|_{L^\infty(0,T;H_0^1(\mathcal{O}^n))} + \left\|f_t^n\right\|_{L^\infty(0,T;L^2(\mathcal{O}^n))} \leqslant C\left(\left\|u_0^n\right\|_{H_0^1(\mathcal{O}^n)}, \left\|u_1^n\right\|_{L^2(\mathcal{O}^n)}, \left\||u_0^n|^2\log|u_0^n|\right\|_{L^1(\mathcal{O}^n)}\right).$$

Now, we can define the approximate solution u^n as follows:

$$u^n = \begin{cases} 0 & \text{for}\quad x\in\mathbb{R}\backslash\mathcal{O}^n,\\ f^n & \text{for}\quad x\in\mathcal{O}^n, \end{cases}$$

where f^n is a weak solution of the problem (5). From proposition 1 follows the estimate

$$\|u^n\|_{L^\infty(0,T;H_0^1(\mathbb{R}))} + \left\|u_t^n\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leqslant C\left(\left\|u_0^n\right\|_{H_0^1(\mathcal{O}^n)}, \left\|u_1^n\right\|_{L^2(\mathcal{O}^n)}, \left\||u_0^n|^2 \log|u_0^n|\right\|_{L^1(\mathcal{O}^n)}\right).$$

Next, we can estimate

$$\left\|u_1^n\right\|_{L^2(\mathcal{O}^n)} \leqslant \left\|u_1^n\right\|_{L^2(\mathbb{R})}.$$

Subsequently, we can write

$$\left\|u_0^n\right\|_{H_0^1(\mathcal{O}^n)}^2 = \int_{\mathcal{O}^n} \left|u_0^n\right|^2 \mathrm{d}x + \int_{\mathcal{O}^n} \left|\nabla u_0^n\right|^2 \mathrm{d}x \leqslant \|u_0\|_{H_0^1(\mathbb{R})}^2 + 2\left(\|u_0\|_{L^\infty(\mathbb{R})} + \|u_0\|_{L^\infty(\mathbb{R})}^3\right)$$
$$\leqslant c\left(\|u_0\|_{H_0^1(\mathbb{R})} + \|u_0\|_{H_0^1(\mathbb{R})}^2 + \|u_0\|_{H_0^1(\mathbb{R})}^3\right),$$

where the Sobolev embedding has been applied. Next,

$$\left\|\left|u_0^n\right|^2 \log\left|u_0^n\right|\right\|_{L^1(\mathcal{O}^n)} = \int_{\mathcal{O}^n} \left|\left|u_0^n\right|^2 \log\left|u_0^n\right|\right| \mathrm{d}x \leqslant \left\||u_0|^2 \log|u_0|\right\|_{L^1(\mathbb{R})}$$
$$+ 2C\|u_0\|_{L^\infty(\mathbb{R})}\left(\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{3}{2}} + \|u_0\|_{L^\infty(\mathbb{R})}^{\frac{5}{2}}\right) \leqslant \left\||u_0|^2 \log|u_0|\right\|_{L^1(\mathbb{R})}$$
$$+ c\left(\|u_0\|_{H^1(\mathbb{R})}^{\frac{5}{2}} + \|u_0\|_{H^1(\mathbb{R})}^{\frac{7}{2}}\right),$$

where we have applied the elementary inequality $|x^2 \log|x|| \leqslant C\left(|x|^{\frac{3}{2}} + |x|^{\frac{5}{2}}\right)$.

Summing up, we have obtained the estimate

$$\|u^n\|_{L^\infty(0,T;H_0^1(\mathbb{R}))} + \left\|u_t^n\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leqslant C.$$

Hence, there exists a subsequence u_n such that

$$u_n \rightharpoonup u \quad \text{weakly - * in} \quad L^\infty(0, T; H^1(\mathbb{R})).$$

Let us now take any $\phi \in C_0^\infty(\mathbb{R} \times (-\infty, T))$. Then there exists l such that $\operatorname{supp}\phi \subset \mathcal{O}^l \times (-\infty, T)$. Since O^l is compact, we obtain by the Aubin–Lions lemma the following convergence:

$$u_n \longrightarrow u \quad \text{strongly in} \quad L^2(0, T; L^2(\mathcal{O}^l)).$$

If we take $n > l$, then we can write

$$\int_0^T \int_{\mathcal{O}^l} u^n(x, t)\phi_{tt}(x, t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{O}^l} u_0^n(x)\phi_t(x, 0)\,\mathrm{d}x - \int_{\mathcal{O}^l} u_1^n(x)\phi(x, 0)\,\mathrm{d}x$$
$$+ \int_0^T \int_{\mathcal{O}^l} \nabla u^n(x, t)\nabla\phi(x, t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathcal{O}^l} u^n(x, t)\phi(x, t)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \varepsilon \int_0^T \int_{\mathcal{O}^l} u^n(x, t) \log|u^n(x, t)|^2\phi(x, t)\,\mathrm{d}x\,\mathrm{d}t.$$

Finally, we can pass to the limits. It finishes the proof.

3. The existence of classical solutions

In this section, we assume that the initial data (u_0, u_1) belong to the space $C^2(\mathbb{R}) \times C^1(\mathbb{R})$. Assuming such kinds of regularity, we will show that the problem (3) has a unique classical solution. Namely, we will prove the following theorem.

Theorem 2. *Let* $u_0 \in C^2(\mathbb{R})$ *and* $u_1 \in C^1(\mathbb{R})$. *Assume that one of the following conditions is fulfilled:*

- *there exists* $\sigma > 0$ *such that* $u_0 \geqslant \sigma$*,*
- *there exists* $\sigma < 0$ *such that* $u_0 \leqslant \sigma$*.*

Then there exists the unique, classical solution to the problem (3). Moreover, the solution depends in a Lipschitz way on the initial data (u_0, u_1).

Proof. We will use the d'Alembert formula and the Duhamel's principle, i.e. the solution to the problem

$$\begin{cases} u_{tt} - u_{xx} = h \\ u(0) = u_0 \\ u_t(0) = u_1, \end{cases}$$

where $h, u_1 \in C^1(\mathbb{R})$, $u_0 \in C^2(\mathbb{R})$, is expressed by the formula

$$u(t,x) = \widetilde{u_0}(t,x) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,\mathrm{d}y + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} h(\tau, y)\,\mathrm{d}y\,\mathrm{d}\tau,$$

where

$$\widetilde{u_0}(t,x) = \tfrac{1}{2}(u_0(x+t) + u_0(x-t)).$$

Let us define the following set:

$$X_\sigma^T = \left\{ u \in C^1((0,T)\times\mathbb{R}) : \|u - \widetilde{u_0}\|_{C^0((0,T)\times\mathbb{R})} \leqslant \frac{\sigma}{2} \right\}.$$

Subsequently, we define the map

$$\mathcal{F} : X_\sigma^T \to C^1((0,T)\times\mathbb{R}), \qquad \mathcal{F}(v) = u,$$

where u is a solution to the linear problem

$$\begin{cases} u_{tt} - u_{xx} = -v - \varepsilon v \log|v|, \\ u(0) = u_0, \\ u_t(0) = u_1. \end{cases}$$

We will show that for a sufficiently small T the map $\mathcal{F}$ is a contraction on X_σ^T:

(i) $\mathcal{F} : X_\sigma^T \to X_\sigma^T$.

From the definition of mapping $\mathcal{F}$,

$$\mathcal{F}(u)(t,x) = \widetilde{u_0}(t,x) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,\mathrm{d}y + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} (-u(\tau,y) - \varepsilon u(\tau,y)\log|u(\tau,y)|)\,\mathrm{d}y\,\mathrm{d}\tau,$$

and from the mean value theorem we have

$$\begin{aligned} \|\mathcal{F}(u) - \widetilde{u_0}\|_{C^0((0,T)\times\mathbb{R})} &\leqslant \frac{1}{2}\left\| \int_{x-t}^{x+t} u_1(y)\,\mathrm{d}y \right\|_{C^0((0,T)\times\mathbb{R})} \\ &\quad + \frac{1}{2}\left\| \int_0^t \int_{x-t+\tau}^{x+t-\tau} (u(\tau,y) + \varepsilon u(\tau,y)\log|u(\tau,y)|)\,\mathrm{d}y\,\mathrm{d}\tau \right\|_{C^0((0,T)\times\mathbb{R})} \\ &\leqslant T\|u_1\|_{C^0(\mathbb{R})} + 2T^2\|u\|_{C^0((0,T)\times\mathbb{R})} \\ &\quad + 2\varepsilon T^2 \left(\|u_0\|_{C^0(\mathbb{R})} + \frac{\sigma}{2} \right) \cdot \left(\sup_{x \in \left[\frac{\sigma}{2}, \|u_0\|_{C^0(\mathbb{R})} + \frac{\sigma}{2}\right]} |\log|x|| \right) \\ &\leqslant T\|u_1\|_{C^0(\mathbb{R})} + 2T^2 \left(\|u_0\|_{C^0(\mathbb{R})} + \frac{\sigma}{2} \right)(1 + \varepsilon C), \end{aligned}$$

where

$$C = \sup_{x\in\left[\frac{\sigma}{2},\|u_0\|_{C^0(\mathbb{R})}+\frac{\sigma}{2}\right]} |\log|x||.$$

By choosing a suitably small T, we obtain

$$\|\mathcal{F}(u) - \widetilde{u_0}\|_{C^0((0,T)\times\mathbb{R})} \leqslant \frac{\sigma}{2}.$$

(ii) We will show that $\mathcal{F}$ is a contraction on X_σ^T. First of all, let us estimate

$$\begin{aligned}\|\mathcal{F}(u) - \mathcal{F}(v)\|_{C^0((0,T)\times\mathbb{R})} &\leqslant \frac{1}{2}\left\|\int_0^t\int_{x-t+\tau}^{x+t-\tau}(u(\tau,y) - v(\tau,y))\,\mathrm{d}\tau\,\mathrm{d}y\right\|_{C^0((0,T)\times\mathbb{R})}\\ &\quad + \frac{1}{2}\varepsilon\left\|\int_0^t\int_{x-t+\tau}^{x+t-\tau}(u(\tau,y)\log|u(\tau,y)| - v(\tau,y)\log|v(\tau,y)|\,\mathrm{d}\tau\,\mathrm{d}y)\right\|_{C^0((0,T)\times\mathbb{R})}\\ &\leqslant 2T^2\|u-v\|_{C^0((0,T)\times\mathbb{R})} + 2T^2\varepsilon(1+C)\|u-v\|_{C^0((0,T)\times\mathbb{R})}\\ &= 2T^2(1+\varepsilon(1+C))\|u-v\|_{C^0((0,T)\times\mathbb{R})}.\end{aligned}$$

So

$$\|\mathcal{F}(u) - \mathcal{F}(v)\|_{C^0((0,T)\times\mathbb{R})} \leqslant 2T^2(1+\varepsilon(1+C))\|u-v\|_{C^0((0,T)\times\mathbb{R})}.$$

Keeping in mind that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t\int_{x-t+\tau}^{x+t-\tau}h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\right) = \int_0^t(h(\tau,x+t-\tau) + h(\tau,x-t+\tau))\,\mathrm{d}\tau,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^t\int_{x-t+\tau}^{x+t-\tau}h(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\right) = \int_0^t(h(\tau,x+t-\tau) - h(\tau,x-t+\tau))\,\mathrm{d}\tau,$$

we obtain

$$\begin{aligned}&\|\partial_t(\mathcal{F}(u) - \mathcal{F}(v))\|_{C^0((0,T)\times\mathbb{R})} + \|\partial_x(\mathcal{F}(u) - \mathcal{F}(v))\|_{C^0((0,T)\times\mathbb{R})}\\ &\quad= \frac{1}{2}\bigg\|\int_0^t u(\tau,x+t-\tau) + \varepsilon u(\tau,x+t-\tau)\log|u(\tau,x+t-\tau)|\\ &\qquad - v(\tau,x+t-\tau) - \varepsilon v(\tau,x+t-\tau)\log|v(\tau,x+t-\tau)|\\ &\qquad + u(\tau,x-t+\tau) + \varepsilon u(\tau,x-t+\tau)\log|u(\tau,x-t+\tau)|\\ &\qquad - v(\tau,x-t+\tau) - \varepsilon v(\tau,x-t+\tau)\log|v(\tau,x-t+\tau)|\,\mathrm{d}\tau\bigg\|_{C^0((0,T)\times\mathbb{R})}\\ &\qquad + \frac{1}{2}\bigg\|\int_0^t(u(\tau,x+t-\tau) + \varepsilon u(\tau,x+t-\tau)\log|u(\tau,x+t-\tau)|)\\ &\qquad - u(\tau,x-t+\tau) - \varepsilon u(\tau,x-t+\tau)\log|u(\tau,x-t+\tau)|\\ &\qquad - v(\tau,x+t-\tau) - \varepsilon v(\tau,x+t-\tau)\log|v(\tau,x+t-\tau)|\\ &\qquad + v(\tau,x-t+\tau) + \varepsilon v(\tau,x+t-\tau)\log|v(\tau,x-t+\tau)|\,\mathrm{d}\tau\bigg\|_{C^0((0,T)\times\mathbb{R})}\\ &\quad\leqslant 2T(1+\varepsilon(1+C))\|u-v\|_{C^0((0,T))\times\mathbb{R})}.\end{aligned}$$

Hence, we obtain

$$\|\mathcal{F}(u) - \mathcal{F}(v)\|_{C^1((0,T)\times\mathbb{R})} \leqslant (1+\varepsilon C)T(T+2)\|u-v\|_{C^0((0,T)\times\mathbb{R})}.$$

Therefore, for a sufficiently small T, we see that $\mathcal{F}$ is a contraction. Hence, from the Banach theorem we obtain the existence of the unique fixed point of the map $\mathcal{F}$.

Let us note that from the formula

$$u(t,x) = \widetilde{u_0}(t,x) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,\mathrm{d}y \\ - \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} (u(\tau,y) + \varepsilon u(\tau,y)\log|u(\tau,y)|)\,\mathrm{d}\tau\,\mathrm{d}y,$$

we see that u is indeed of C^2-class. This finishes the proof of the existence of the solutions. We will now focus on showing the continuous dependence of the solutions on initial data. Let us assume that u^1 and u^2 are the solutions to the following problems:

$$\begin{cases} u^i_{tt} - u^i_{xx} = -u^i - \varepsilon u^i \log|u^i|, \\ u^i(0) = u^i_0, \\ u^i_t(0) = u^i_1, \end{cases}$$

for $i = 1, 2$. Hence, $w = u^1 - u^2$ fulfils the equation

$$\begin{cases} w_{tt} - w_{xx} = -w - \varepsilon u^1 \log|u^1| + \varepsilon u^2 \log|u^2|, \\ w(0) = w_0 = u^1_0 - u^2_0, \\ w_t(0) = w_1 = u^1_1 - u^2_1. \end{cases} \tag{6}$$

The solution to the problem (6) can be expressed by the formula

$$w(t,x) = \widetilde{w_0}(t,x) + \frac{1}{2}\int_{x-t}^{x+t} w_1(y)\,\mathrm{d}y \\ + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} (-w - \varepsilon u^1 \log|u^1| + \varepsilon u^2 \log|u^2|)\,\mathrm{d}\tau\,\mathrm{d}y.$$

Therefore, we obtain

$$\|w\|_{C^0((0,T)\times\mathbb{R})} \leqslant \|w_0\|_{C^0(\mathbb{R})} + T\|w_1\|_{C^0(\mathbb{R})} + 2T^2(1 + \varepsilon(1 + C))\|w\|_{C^0((0,T)\times\mathbb{R})}.$$

Hence, for a sufficiently small T, we obtain

$$\|w\|_{C^0((0,T)\times\mathbb{R})} \leqslant C_1(\|w_0\|_{C^0(\mathbb{R})} + \|w_1\|_{C^0(\mathbb{R})}).$$

Subsequently, it can be shown that

$$\|\partial_t w\|_{C^0((0,T)\times\mathbb{R})} + \|\partial_x w\|_{C^0((0,T)\times\mathbb{R})} \\ \leqslant 2(\|w_0\|_{C^0(\mathbb{R})} + \|w_1\|_{C^0((0,T)\times\mathbb{R})} + C_2\|w\|_{C^0((0,T)\times\mathbb{R})}),$$

$$\|\partial_t\partial_t w\|_{C^0((0,T)\times\mathbb{R})} + \|\partial_x\partial_t w\|_{C^0((0,T)\times\mathbb{R})} + \|\partial_x\partial_x w\|_{C^0((0,T)\times\mathbb{R})} \\ \leqslant 3(\|w_0\|_{C^0(\mathbb{R})} + \|w_1\|_{C^0(\mathbb{R})} + C_3(\|\partial_x w\|_{C^0((0,T)\times\mathbb{R})} + \|w\|_{C^0((0,T)\times\mathbb{R})})).$$

Therefore,

$$\|u^1 - u^2\|_{C^2((0,T)\times\mathbb{R})} \leqslant \widetilde{C}\left(\left\|u^1_0 - u^2_0\right\|_{C^2(\mathbb{R})} + \left\|u^1_1 - u^2_1\right\|_{C^1(\mathbb{R})}\right),$$

which finishes the proof of the theorem.

□

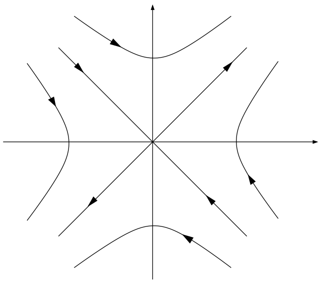

Figure 1. $\beta = 0$.

4. Traveling waves

In this section, we shall show the existence of traveling waves of the problem (2). Let us assume that the solution to the problem (2) has the following form:

$$u(x, t) = f(x - vt).$$

Then

$$f''(1 - v^2) = f + \varepsilon f \log|f|^2.$$

Let us introduce the standard notation

$$\gamma = \frac{1}{\sqrt{1 - v^2}}, \qquad \beta = \gamma^2 \varepsilon.$$

Hence, we can write

$$f'' = \gamma^2 f + \beta f \log|f|^2. \tag{7}$$

If we put $g = f'$, then (7) takes the form

$$\begin{cases} f' = g, \\ g' = \gamma^2 f + \beta f \log|f|^2. \end{cases} \tag{8}$$

Let us denote

$$F = \begin{pmatrix} g \\ \gamma^2 f + \beta f \log|f|^2 \end{pmatrix}.$$

Observe that (8) is a Hamiltonian system, namely

$$\operatorname{div} F = 0.$$

We divide our analysis into three cases.

4.1. Case $\beta = 0$

In the case $\beta = 0$, we obtain the linear system

$$\begin{cases} f' = g, \\ g' = \gamma^2 f. \end{cases}$$

Hence, we can draw the phase portrait presented in figure 1.

4.2. Case $\beta > 0$

In this subsection, we shall work with the system (8) with $\beta > 0$. First of all, let us see that the field F vanishes at

$$(x_-, 0) = \left(-\mathrm{e}^{-\frac{\gamma^2}{2\beta}}, 0\right), \quad (x_+, 0) = \left(\mathrm{e}^{-\frac{\gamma^2}{2\beta}}, 0\right), \quad (0, 0).$$

The points $(x_+, 0)$, $(x_-, 0)$ are the hyperbolic one. Hence, according to the Grobman–Hartman theorem, we can linearize our system around these points. By linearizing one gets

$$DF = \begin{pmatrix} 0 & 1 \\ \gamma^2 + 2\beta + \beta \log|f|^2 & 0 \end{pmatrix}$$

and

$$(DF)(x_+, 0) = (DF)(x_-, 0) = \begin{pmatrix} 0 & 1 \\ 2\beta & 0 \end{pmatrix}.$$

Now, we can state

Theorem 3. *For each v such that $|v| < 1$, there exists a trajectory joining the points $(x_+, 0)$ and $(x_-, 0)$.*

Proof. Let us find the Hamiltonian, i.e. H such that

$$\begin{cases} f' = \dfrac{\partial H}{\partial g}, \\ g' = -\dfrac{\partial H}{\partial f}. \end{cases}$$

But according to (8)

$$\frac{\partial H}{\partial g} = g,$$

which implies

$$H(g, f) = \frac{g^2}{2} + C(f) - \frac{\partial H}{\partial f} = \gamma^2 f + \beta f \log|f|^2.$$

Hence,

$$H(g, f) = \frac{g^2}{2} + \frac{f^2}{2}(\beta - \gamma^2) - \beta f^2 \log|f|.$$

Now, let us consider the level sets of H:

$$H(g, f) = \frac{g^2}{2} + \frac{f^2}{2}(\beta - \gamma^2) - \beta f^2 \log|f| = c,$$

where $c \in \mathbb{R}$.

Next, let us take a special value of c. Namely, let us put $c = K$, where K is defined as follows:

$$K = \tfrac{1}{2}({x_-}^2(\beta - \gamma^2) - \beta {x_-}^2 \log|x_-|^2) = \tfrac{1}{2}({x_+}^2(\beta - \gamma^2) - \beta {x_+}^2 \log|x_+|^2) = \tfrac{1}{2}\beta\, \mathrm{e}^{-\frac{\gamma^2}{\beta}}.$$

Then for $f \in (x_-, x_+)$

$$g(f) = \sqrt{2K - f^2(\beta - \gamma^2) + \beta f^2 \log|f|^2}$$

is well defined, since

$$2K - f^2(\beta - \gamma^2) + \beta f^2 \log|f|^2 > 0. \tag{9}$$

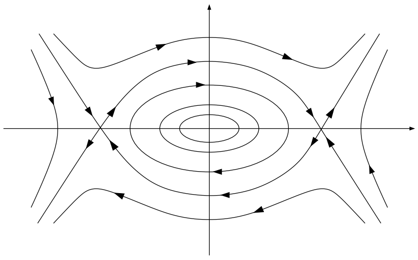

Figure 2. $\beta > 0$.

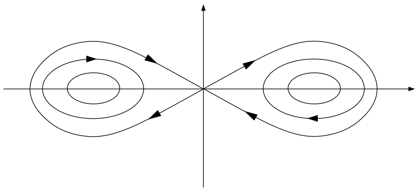

Figure 3. $\beta < 0$.

Indeed, for $f \in [0, x_+)$ we consider a function

$$U(f) = 2K - f^2(\beta - \gamma^2) + \beta f^2 \log|f|^2.$$

Next, it is easy to see that

$$U(x_+) = 0, \qquad U(0) = 2K > 0.$$

Moreover,

$$U'(f) = 2f(\gamma^2 + \beta \log|f|^2) \leqslant 2x_+(\gamma^2 + \beta \log|x_+|^2) = 0.$$

This proves (9) and finishes the proof. □

Using similar techniques as in the proof of the above theorem and the Morse lemma (see [22]) around the point (0, 0), one can draw the phase portrait for $\beta > 0$ (see figure 2).

4.3. Case $\beta < 0$

Processed in the same manner as in the previous subsection, we obtain the picture that is presented in figure 3.

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